EPFL - Fall semester 2025 J. Scherer
Cohomology rings **Solutions 2**
MATHEMATICS

◊ Exercice 1. The Hurewicz homomorphism.

Let (X, x_0) be a pointed and path-connected space. In this exercise you will construct a map

$$\mathrm{Hu}\colon \pi_1(X) \to H_1(X;\mathbb{Z}).$$

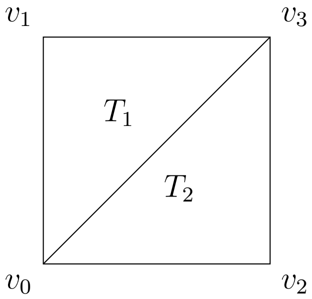
1. Explain that a loop ω in X, based at x_0, can be lifted to a map $\tilde{\omega}\colon \Delta^1 \approx I \to X$ and check that $\tilde{\omega}$ is a cycle in $C_1^{sing}(X;\mathbb{Z})$.
2. Show that $\mathrm{Hu}([\omega]) := [\tilde{\omega}]$ is well-defined in $H_1(X;\mathbb{Z})$.
3. Show that Hu is a homomorphism.
4. Check that Hu is a natural transformation of functors, which you explicit.

Solution 1. (shorten version of the hand-in by Annaëlle and Leila)

1. A loop $w : S^1 \to X$ can be lifted to a $1-$simplex via the composition with the quotient map $q\colon I \to S^1 \simeq I/\{0,1\}$. Then $\tilde{w} = q \circ w$ is clearly a cycle in $C_1(X)$.
2. The lift $\tilde{\omega}$ is a cycle in $C_1(X;\mathbb{Z})$ so it defines a class in $H_1(X;\mathbb{Z})$.
   Let α and γ be homotopic loops in X. There exists a pointed homotopy :

$$\begin{aligned} H : I \times I &\longrightarrow X \\ (s,0) &\longmapsto \alpha(s) \\ (s,1) &\longmapsto \gamma(s) \\ (0,t) &\longmapsto x_0 \\ (1,t) &\longmapsto x_0 \end{aligned}$$

Let us take the square $I \times I$ and label its vertices with v_0, v_1, v_2, v_3 as in the diagram below. We split it along the diagonal v_0, v_2, forming two triangles v_0, v_1, v_3 and v_0, v_2, v_3, that we will denote respectively T_1 and T_2, and that are clearly homeomorphic to the 2-simplex Δ^2.



Let us define the two following chains $\sigma_1, \sigma_2 \in C_2^{sing}(X;\mathbb{Z})$:

$$\begin{aligned} \sigma_1 : \Delta^2 \approx T_1 &\longrightarrow X \\ (s,t) &\longmapsto H(s,t) \\ \sigma_2 : \Delta^2 \approx T_2 &\longrightarrow X \\ (s,t) &\longmapsto H(s,t) \end{aligned}$$

Let us now take the image with the boundary map :

$$\begin{aligned} \partial\sigma_1 &= \sigma_1|_{[v_1,v_3]} - \sigma_1|_{[v_0,v_3]} + \sigma_1|_{[v_0,v_1]} = H_1 - D + h_0 \\ \partial\sigma_2 &= \sigma_2|_{[v_2,v_3]} - \sigma_2|_{[v_0,v_3]} + \sigma_2|_{[v_0,v_2]} = h_1 - D + H_0 \end{aligned}$$

where the maps above denote :

$$\begin{aligned} H_t : \Delta^1 &\longrightarrow X \\ s &\longmapsto H(s,t) \end{aligned} \qquad\qquad \begin{aligned} D : \Delta^1 &\longrightarrow X \\ t &\longmapsto H(t,t) \end{aligned} \qquad\qquad \begin{aligned} h_s : \Delta^1 &\longrightarrow X \\ t &\longmapsto H(s,t) \end{aligned}$$

So we obtain that $H_0 = \tilde{\alpha}$, $H_1 = \tilde{\gamma}$ and both h_0 and h_1 are constant functions in x_0 so we can conclude :

$$\partial(\sigma_1 - \sigma_2) = H_1 - D + h_0 - h_1 + D - H_0 = H_1 - H_0 = \tilde{\gamma} - \tilde{\alpha}$$

This implies that $[\tilde{\gamma} - \tilde{\alpha}] = [\tilde{\gamma}] - [\tilde{\alpha}] = 0 \in H_1(X, \mathbb{Z})$ so indeed $\mathrm{Hu}([\alpha]) = \mathrm{Hu}([\gamma])$.

3. We need to prove $[\widetilde{w \star z}] = [\tilde{z}] + [\tilde{w}]$, where $\star$ is the concatenation of paths. Let's find a chain $\sigma : \Delta^2 \to X$ with boundary $\partial_2(\sigma) = \tilde{z} + \tilde{w} - \widetilde{w \star z}$.
   First note that we can see $\widetilde{w \star z}$ as the following 1-chain $\widetilde{w \star z} : \Delta^1 \approx I \to X$

   $$w \tilde{\star} z(t) = \begin{cases} w(2t) & \text{si } 0 \leq t \leq \frac{1}{2} \\ z(2t-1) & \text{si } \frac{1}{2} < t \leq 1 \end{cases}$$

   Then we construct σ by taking first the projection p on the edge $[v_0, v_2]$ (if we write $\Delta^2 = [v_0, v_1, v_2]$) and then applying $\widetilde{w \star z}$ on $I \approx [v_0, v_2]$:

   $$\sigma : \Delta^2 \xrightarrow{p} [v_0, v_2] \approx I \xrightarrow{\widetilde{w \star z}} X$$

   Under p, the image of $[v_0, v_1]$ corresponds to the first half of $[v_0, v_2] \approx I$, and $[v_1, v_2]$ to the second one. Then composing with $\widetilde{w \star z}$ we have

   $$\sigma|_{[v_0,v_1]} = \tilde{w}, \quad \sigma|_{[v_1,v_2]} = \tilde{z}, \quad \sigma|_{[v_0,v_2]} = \widetilde{w \star z}$$

   and finally :

   $$\partial_2\sigma = \tilde{z} - \widetilde{w \star z} + \tilde{w}.$$

4. We need to check that for any $(X, x_0), (Y, y_0) \in \mathrm{Top}^*$ and any morphism $f \in Top^*((X, x_0), (Y, y_0))$ the following diagram commutes :

   $$\begin{tikzcd} \pi_1(X) \arrow[r, "\mathrm{Hu}_X"] \arrow[d, "\pi_1 f"'] & H_1(X, \mathbb{Z}) \arrow[d, "H_1 f"] \\ \pi_1(Y) \arrow[r, "\mathrm{Hu}_Y"'] & H_1(Y, \mathbb{Z}) \end{tikzcd}$$

   Let us choose any $[\omega] \in \pi_1(X)$ then we see that :

   $$H_1 f \circ \mathrm{Hu}_X([\omega]) = H_1 f([\tilde{\omega}]) = [f \circ \tilde{\omega}] = [f \,\tilde{\circ}\, \omega] = \mathrm{Hu}_Y([f \circ \omega]) = \mathrm{Hu}_Y \circ \pi_1 f([\omega])$$

   Since the diagram commutes we can conclude that Hu is a well defined natural transformation.

$\Diamond$ Exercice 2. The Hurewicz Theorem.

We keep the notations of Exercice 1. The goal is to show that Hu induces an isomorphism

$$(\pi_1 X)_{ab} \cong H_1(X;\mathbb{Z}).$$

1. Let α be a path in X from x_0 to x and γ a loop based at x. Show that the loops $\alpha \star \gamma \star \overline{\alpha}$ and γ induce the same class in $H_1(X;\mathbb{Z})$ via Hu.
2. Show that Hu is surjective.
3. Prove that Hu induces an isomorphism as desired when X is a wedge of circles.
4. Prove that Hu induces an isomorphism as desired when X is a 2-dimensional CW-complex.
5. Prove that the general statement follows from the case of a 2-dimensional CW-complex.

Solution 2 (by Léo and Milo)

1. This is clear from 1.3.
2. Let $\alpha = \sum\limits_i n_i\ \alpha_i \in C_1^{sing}(X;\mathbb{Z})$ be a cycle. Upon repeating some elements of the sum many times, we may assume that $n_i = \pm$ for every i. Furthermore upon swapping out α_i with $\overline{\alpha_i}$, we may assume that every n_i is 1, and may hence write $\alpha = \sum\limits_i \alpha_i$. Let i be such that α_i is not a loop. Since α is a cycle, there must be some j such that $\alpha_i(1) = \alpha_j(0)$. By the claim, we may hence replace $\alpha_i + \alpha_j$ with $\alpha_i \star \alpha_j$. If $\alpha_j(1) \neq \alpha_i(0)$, we may reiterate the same reasoning until we get a loop. Hence we may assume that all σ_i are loops. By the previous point, we may furthermore assume that they all have the same basepoint x_0. Hence taking the concatenation (in any order) of all α_i, we get a loop γ, such that $\mathrm{Hu}([\gamma]) = [\sigma]$, which proves the surjectivity, as desired.
3. Let $X = \bigvee\limits_{\alpha \in A} S^1_\alpha$ be a wedge of circles. It is well-known that the fundamental group of X is the free group generated by A, which we denote F_A, and that $H_1(X) \cong \bigoplus\limits_A \mathbb{Z}$ is the free abelian group. We know it to be the abelianization of F_A. In our case, Hu sends any generator $a \in A$ of F_A to its corresponding copy of $\mathbb{Z}$ in $\bigoplus\limits_A \mathbb{Z}$, which correspond to the map from F_A into its abelianization, i.e one has a commutative square

$$\begin{tikzcd}
\pi_1(X) \arrow[r, "\mathrm{Hu}"] \arrow[d, "\cong"] & H_1(X) \arrow[d, "\cong"] \\
F_A \arrow[r] & (F_A)_{ab}
\end{tikzcd}$$

We hence have

$$\left(\pi_1\left(\bigvee_{\alpha \in A} S^1_\alpha\right)\right)_{ab} \cong (F_A)_{ab} \overset{\mathrm{Hu}}{\cong} \bigoplus_A \mathbb{Z} \cong H_1\left(\bigvee_{\alpha \in A} S^1_\alpha\right)$$

as desired.

4. Let X be a CW-complex of dimension 2. By assumption X is path-connected so its 1-skeleton is homotopic to a wedge of circles $X^{(1)} = \bigvee_\alpha S^1_\alpha$, and the 2-cells are attached by maps

$f_\beta : S^1_\beta \to \bigvee_\alpha S^1_\alpha$. We denote by $f : \bigvee_\beta S^1_\beta \to \bigvee_\alpha S^1_\alpha$ the wedge of the attaching maps i.e. X is the pushout of the diagram

$$\begin{tikzcd}
\bigvee_\beta D^2_\beta & \bigvee_\beta S^1_\beta \arrow[l, hook'] \arrow[r, "f"] & \bigvee_\alpha S^1_\alpha
\end{tikzcd}$$

By Seifert-Van Kampen, $\pi_1(X)$ is isomorphic to the quotient $\frac{\pi_1(\bigvee_\alpha S^1_\alpha)}{N}$, where N is the normal subgroup generated by the image of $f_* : \pi_1(\bigvee_\beta S^1_\beta) \to \pi_1(\bigvee_\alpha S^1_\alpha)$. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, hook] & \pi_1(\bigvee_\alpha S^1_\alpha) \arrow[r, two heads, "i_*"] \arrow[d, two heads, "\mathrm{Hu}"] & \pi_1(X) \arrow[r] \arrow[d, two heads, "\mathrm{Hu}"] & 0 \\
H_1(\bigvee_\beta S^1_\beta) \arrow[rr, "f_*"] & & H_1(\bigvee_\alpha S^1_\alpha) \arrow[r, "i_*"] & H_1(X) &
\end{tikzcd}$$

where the first row is exact as explained, and the second row is exact by long exact sequence in homology. Let $\alpha \in \ker(\mathrm{Hu}) \subset \pi_1(X)$, we want to show that α is in the commutator subgroup of $\pi_1(X)$. Let $\beta \in \pi_1(\bigvee_\alpha S^1_\alpha)$ be a preimage of α, then $i_*\mathrm{Hu}(\beta) = 0$ by commutativity of the diagram, so $\mathrm{Hu}(\beta) \in \ker(i_*) = \mathrm{im}(f_*)$, and we can choose some $\gamma \in H_1(\bigvee_\beta S^1_\beta)$ such that $f_*(\gamma) = \mathrm{Hu}(\beta)$. Moreover, by surjectivity of the Hurewicz morphism, there exists some $\omega \in \pi_1(\bigvee_\beta S^1_\beta)$ such that $\mathrm{Hu}(\omega) = \gamma$. Since Hu is a natural transformation, one has the following commutative diagram

$$\begin{tikzcd}
\pi_1(\bigvee_\beta S^1_\beta) \arrow[r, "f_*"] \arrow[d, two heads, "\mathrm{Hu}"] & \pi_1(\bigvee_\alpha S^1_\alpha) \arrow[r, two heads, "i_*"] \arrow[d, two heads, "\mathrm{Hu}"] & \pi_1(X) \arrow[d, two heads, "\mathrm{Hu}"] \\
H_1(\bigvee_\beta S^1_\beta) \arrow[r, "f_*"] & H_1(\bigvee_\alpha S^1_\alpha) \arrow[r, "i_*"] & H_1(X)
\end{tikzcd}$$

In particular $\mathrm{Hu}(\beta) = f_*\mathrm{Hu}(\omega) = \mathrm{Hu}f_*(\omega)$, i.e. $\beta - f_*(\omega)$ is in the kernel of Hu: $\pi_1(\bigvee_\alpha S^1_\alpha) \to H_1(\bigvee_\alpha S^1_\alpha)$ which is exactly the commutator subgroup of $\pi_1(\bigvee_\alpha S^1_\alpha)$ by the previous point. By exactness of the first row of the first diagram, we obtain $i_*(\beta - f_*(\omega)) = \alpha$, in other words, α belongs to the image of the commutator group, so it has to belong to $[\pi_1(X), \pi_1(X)]$ because group homomorphisms send commutators to commutators. This shows that $\ker(\mathrm{Hu}) \subset [\pi_1 X, \pi_1 X]$, and that we conclude that $H_1(X, \mathbb{Z}) \cong \pi_1(X)_{ab}$, since the other inclusion follows directly from the fact that homology groups are abelian.

5. From Chapter 2, Lemma 2.34 of Hatcher, we have the following lemma : If X is a *CW* complex, then the map $H_k(X^n) \to H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ of the n-skeleton of X is an isomorphism for $k < n$. Let X be any CW complex. By the lemma above one has $H_1(X) \cong H_1(X^2)$. By the previous point, we know that the statement holds for 2-dimensional CW complex. It follows that the statement holds in the general case.

   Note that this statement can easily be generalized to any path-connected space using the CW approximation theorem. This theorem states that any space X is weak homotopy equivalent to a CW complex $Y \xrightarrow{\sim} X$. Since weak homotopy equivalences induce isomorphisms in homotopy and homology, we obtain the desired statement.

$\Diamond$ **Exercice 3. The deficiency of a group.** (by Grégoire and Maxime)
Let G be a finitely presented group. The deficiency $\mathsf{def}\ G \in \mathbb{Z}$ is the largest difference $|S| - |R|$ between the number of generators S and that of relators R in a presentation of the group G. The

aim of this exercise is to show that the deficiency is bounded by the difference of ranks of the first and second homology groups of G.

1. Let A be a finitely generated abelian group. Recall the definition of the integral rank $\mathsf{rank}_{\mathbb{Z}}(A)$ and show that it is equal to the rational rank of the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$.
   Hint : Recall the fundamental theorem of finitely generated abelian groups.
2. Given a finite presentation $\langle S \mid R\rangle$ of G, let N be the normal subgroup in $F(S)$ generated by R. Show that $\mathsf{rank}_{\mathbb{Z}}\left((H_1(N;\mathbb{Z})_G\right) \leq |R|$.
3. Prove then that $|S| - |R| \leq \mathsf{rank}_{\mathbb{Z}} H_1(G;\mathbb{Z}) - \mathsf{rank}_{\mathbb{Z}} H_2(G;\mathbb{Z})$.
4. Conclude that for any finitely presented group G we have $\mathsf{def}\ G \leq \mathsf{rank}_{\mathbb{Z}} H_1(G;\mathbb{Z}) - \mathsf{rank}_{\mathbb{Z}} H_2(G;\mathbb{Z})$.

Solution 3.

1. The fundamental theorem of finitely generated abelian groups states that A is of the form :

   $$A \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z} \oplus \bigoplus_{i=1}^{k} \mathbb{Z}$$

   for some $n_1, ...n_r \in \mathbb{Z}^{\geq 2}$ and $r, k \in \mathbb{N}$. Recall that the integral rank of an abelian group is the size of a maximal linearly independent subset, or equivalently, the rank of its torsion-free part (i.e. k in our case).
   Applying $(-) \otimes_{\mathbb{Z}} \mathbb{Q}$, we get

   $$\begin{aligned} A \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \left(\mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z} \oplus \bigoplus_{i=1}^{k} \mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \\ &\cong \underbrace{(\mathbb{Z}/n_1\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q})}_{=0} \oplus \cdots \oplus \underbrace{(\mathbb{Z}/n_r\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q})}_{=0} \oplus \bigoplus_{i=1}^{k} \underbrace{(\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q})}_{=\mathbb{Q}} \\ &\cong \bigoplus_{i=1}^{k} \mathbb{Q}, \end{aligned}$$

   which has dimension $\mathsf{dim}_{\mathbb{Q}} = k = \mathsf{rank}_{\mathbb{Z}}(A)$ as a $\mathbb{Q}$-vector-space.

2. Let $G = \langle S \mid R\rangle$ be a presentation of G, and let $H = \langle R\rangle \leq F(S)$ be the subgroup generated by R. Recall that $G = F(S)/N$, where N is the normal subgroup of $F(S)$ generated by R. Hence, we can write N as the union of the conjugacy classes of H, that is $N = \bigcup_{w\in F(S)} wHw^{-1}$, which is generated by the elements of the form $\{wrw^{-1} : w \in F(S), r \in R\}$.

   By Proposition 8.1 from the lecture, we have that $H_1(N,\mathbb{Z}) = N_{ab} = N/[N,N]$. We now define an action $F(S) \curvearrowright N$ by $w \cdot n = wnw^{-1}$ for any $w \in F(S)$. Observe that the action is well-defined since $wnw^{-1} \in N$ by normality of N. Moreover, this action induces an action $G \curvearrowright N_{ab}$ as N acts trivially on N_{ab} by conjugation.

   N being generated by the elements $\{wrw^{-1} : r \in R, w \in F(S)\}$, it follows that N_{ab} is generated by their equivalence classes in the quotient. Taking the G-invariants, we identify $[w][r][w]^{-1} = [wrw^{-1}] \sim [r]$ for any $[w] \in G$. In particular, this implies that $(N_{ab})_G$

is generated by the elements of the form $\{[r] : r \in R\}$. Using item **(1)**, we obtain that $\mathsf{rank}_{\mathbb{Z}} H_1(N, \mathbb{Z})_G = \mathsf{rank}_{\mathbb{Z}} \left(N_{ab}\right)_G = \mathsf{dim}_{\mathbb{Q}} \left(\left(N_{ab}\right)_G \otimes_{\mathbb{Z}} \mathbb{Q}\right)$. The latter being generated by the elements $[r] \otimes 1$, for $r \in R$, as a $\mathbb{Q}$-vector-space, its dimension cannot be greater than $|R|$. Therefore, we finally get the desired inequality : $\mathsf{rank}_{\mathbb{Z}} H_1(N, \mathbb{Z})_G \leq |R|$.

3. We use the exact sequence given in the proposition 9.2 of the lecture :

$$0 \longrightarrow H_2(G; \mathbb{Z}) \longrightarrow H_1(N; \mathbb{Z})_G \longrightarrow H_1(F(S); \mathbb{Z}) \longrightarrow H_1(G; \mathbb{Z}) \longrightarrow 0.$$

We also use the alternating sum of ranks for finitely generated abelian groups property : if

$$0 \longrightarrow A_n \longrightarrow \cdots \longrightarrow A_0 \longrightarrow 0$$

is an exact sequence of finitely generated abelian groups, then

$$\sum_{i=0}^{n} (-1)^i \mathsf{rank}_{\mathbb{Z}} A_i = 0.$$

Applying this to the above exact sequence, we get that

$$\mathsf{rk} H_1(G; \mathbb{Z}) - \mathsf{rk} H_2(G; \mathbb{Z}) = \mathsf{rk} H_1(F(S); \mathbb{Z}) - \mathsf{rk} H_1(N; \mathbb{Z})_G.$$

Now, using that $H_1(F(S); \mathbb{Z}) \cong F(S)_{ab} \cong \bigoplus_{s \in S} \mathbb{Z}$, and therefore $\mathsf{rk} H_1(F(S), \mathbb{Z}) = |S|$, and the result of item **(2)**, we obtain the desired inequality :

$$\mathsf{rk} H_1(G; \mathbb{Z}) - \mathsf{rk} H_2(G; \mathbb{Z}) \geq |S| - |R|.$$

4. As this is true for any finite presentation $\langle S \mid R \rangle$ of G and the bound does not depend on the choice of S and R, it follows that the inequality still holds for the supremum of $|S| - |R|$ over all finite presentations. And therefore, we conclude that :

$$\mathsf{def} G \leq \mathsf{rk} H_1(G; \mathbb{Z}) - \mathsf{rk} H_2(G; \mathbb{Z}).$$

◊ **Exercice 4. Whitehead's Theorem for $K(G, 1)$'s.** We will see here that under certain circumstances the homotopy pushout of $K(\pi, 1)$'s is again a $K(\pi, 1)$. We consider two injective group homomorphisms $\varphi_1 \colon H \hookrightarrow G_1$ and $\varphi_2 \colon H \hookrightarrow G_2$. We will assume that the normal form theorem for amalgamted sums of groups is known, in particular G_1 and G_2 can be seen as subgroups of $G_1 *_H G_2$ via the canonical maps.

1. Let X' be a connected sub-CW-complex of the connected CW-complex X such that $\pi_1 X' \to \pi_1 X$ is injective. If $p \colon \widetilde{X} \to X$ is the universal cover, show that each connected component of $p^{-1}(X')$ is a universal cover of X'.
2. Give an example of the previous situation where $p^{-1}(X')$ is connected, but not equal to $\widetilde{X}$, and one when this preimage is not connected.
3. Give a counter-example when the map on fundamental groups is not injective.
4. Prove that there exist Eilenberg-Mac Lane spaces $K(H, 1)$, $K(G_1, 1)$, $K(G_2, 1)$ and maps $f_1 \colon K(H, 1) \to K(G_1, 1)$ and $f_2 \colon K(H, 1) \to K(G_2, 1)$ inducing φ_1 and φ_2 on π_1.

5. Prove the Whitehead Theorem : The homotopy pushout of f_1 and f_2 is an Eilenberg-Mac Lane space $K(G_1 *_H G_2, 1)$.

Solution 4 (by Louis and Leo)

1. Let C be a connected component of $p^{-1}(X')$. To show that $q := p|_C : C \to X'$ is a universal cover, we need to show two different points.
   (a) <u>$q : C \to X'$ is a covering map :</u> More precisely, we need to show that $\forall x' \in X', \exists U_{x'} \subset X'$ with $x' \in U_{x'}$ such that
   $$q^{-1}(U_{x'}) = \coprod_{\alpha \in A} V_\alpha \subset C, \quad A \text{ is a discrete set}$$
   and such that
   $$q|_{V_\alpha} : V_\alpha \to U_{x'} \text{ is a homeomorphism.}$$
   This follows mostly from the fact that p is a covering map itself. That is, for any $x' \in X' \subset X$, there exists $U_{x'} \subset X$ a neighbourhood of x' such that
   $$p^{-1}(U_{x'}) = \coprod_{\alpha \in A} V_\alpha \subset \widetilde{X}, \quad A \text{ is a discrete set}$$
   and such that
   $$p|_{V_\alpha} : V_\alpha \to U_{x'} \text{ is a homeomorphism.}$$
   Replacing $U_{x'}$ by $U_{x'} \cap X'$ and V_α by $V_\alpha \cap C$, it follows that $q = p|_C$ is indeed a covering map.
   (b) <u>$\pi_1(C) \cong 0$:</u> Consider the commuting diagram
   $$\begin{tikzcd} & \widetilde{X} \arrow[dr, "p"] & \\ C \arrow[ur, hook, "j"] \arrow[r, "q"'] & X' \arrow[r, hook, "i"'] & X \end{tikzcd}$$
   Functoriality of π_1 yields the commuting diagram
   $$\begin{tikzcd} & 0 \arrow[dr, "p_*"] & \\ \pi_1(C) \arrow[ur, "j_*"] \arrow[r, hook, "q_*"'] & \pi_1(X') \arrow[r, hook, "i_*"'] & \pi_1(X) \end{tikzcd}$$
   where we used : X' is a **universal** cover ; a covering induces an injection on homotopy groups ; the hypothesis $\pi_1(X') \hookrightarrow \pi_1(X)$. Finally, since $i_* \circ q_* = p_* \circ j_* = 0$ and i_* and q_* are injective, we must have $\pi_1(C) = 0$.
2. — Example of $p^{-1}(X')$ connected but not equal to $\widetilde{X}$:
   $$\widetilde{X} = \mathbb{R}^2, \quad X = S^1 \times \mathbb{R}, \quad X' = S^1 \times \{0\} \Rightarrow p^{-1}(X') = \mathbb{R} \times \{0\}$$
   with
   $$\pi_1 X' = \mathbb{Z} \hookrightarrow \mathbb{Z} = \pi_1 X.$$

   — Example of $p^{-1}(X)$ disconnected :

$$\widetilde{X} = \text{CayleyGraph}(F(a,b)), \quad X = S^1_a \vee S^1_b, \quad X' = S^1_a \Rightarrow p^{-1}(X') \text{ disconnected}$$

   with

$$\pi_1 X' = F(a) \hookrightarrow F(a,b) = \pi_1 X.$$

3. We take the example

$$\widetilde{X} = X = \mathbb{R}^2, \quad X' = S^1.$$

   The induced map on fundamental groups is injective and the proposition does not hold :

$$\pi_1(X') \cong \mathbb{Z} \not\to 0 \cong \pi_1(X) \text{ and } p : S^1 \to S^1 \text{ is not the universal covering}$$

4. First note that BH is a $K(H,1)$ and we will now use BH and BG instead of $K(H,1)$ and $K(G,1)$. Let $\varphi : H \to G$ be a group homomorphism. We will show that it induces the map $f : BH \to BG$ inducing ϕ itself on π_1. First $\varphi : H \to G$ induces a map $H^{n+1} \to G^{n+1}$ so it maps the n-cells of EH to those of EG. Since the gluing of the cells is of combinatorial nature, the gluing maps commute with the maps induced by φ. So we get a continuous map $\bar{\varphi} : EH \to EG$ sending the n-cell in EH labelled $(h_0, \ldots, h_n)$ to the n-cell in G labelled $(\varphi(h_0), \ldots, \phi(h_n))$.
   Since $BH = EH/H$ is a quotient of EH, consider the n-cells labelled $(h_i)_{i=0}^n$ and $(hh_i)_{i=0}^n$ in EH. They are sent to the n-cells labelled $(\varphi(h_i))_i$ and $(\varphi(h)\varphi(h_i))_i$ in EG which get identified in $BG = EG/G$. So, $\bar{\varphi}$ actually induces a map $f : BH \to BG$.

$$\begin{tikzcd}
EH \arrow[r, "\bar{\varphi}"] \arrow[d, "p"] & EG \arrow[d, "q"] \\
BH \arrow[r, "f"] & BG
\end{tikzcd}$$

   Since the action of H on EH is totally discontinuous and EH is simply connected, we have $\pi_1(EH/H) \cong H$. Let us explicit a bit more this isomorphism. Pick a base point e_0 in EH and for each $h \in H$ choose a path $\gamma_h : e_0 \to h \cdot e_0$ in EH. Note that γ_h is unique up to homotopy by contractibility of EH. We then have the isomophism ι^H

$$\begin{aligned} \iota^H : H &\cong \pi_1(BH) \\ h &\mapsto [p \circ \gamma_h] \end{aligned}$$

   We define the same objects for G. We want to show that $f_* : \pi_1(BH) \to \pi_1(BG)$ is φ. More precisely we want to find $f_* \circ \iota^H = \iota^G \circ \varphi$. Let $h \in H$, we have that

$$f_*(\iota^H(h)) \overset{def}{=} [f \circ p \circ \gamma_h] \underset{\text{square}}{\overset{\text{above}}{=}} [q \circ \bar{\varphi} \circ \gamma_h] \overset{EG \simeq *}{=} [q \circ \gamma_{\varphi(h)}] \overset{def}{=} \iota^G(\varphi(h))$$

   So f induces φ on π_1.

5. Our model for the homotopy pushout of $K(G_1,1) \xleftarrow{f_1} K(H,1) \xrightarrow{f_2} K(G_2,1)$ will be

$$M = M(f_1, f_2) = K(G_1,1) \sqcup K(H,1) \times I \sqcup K(G_2,1) \Big/ {\substack{(x,0) \sim f_1(x) \\ (x,1) \sim f_2(x)}}$$

   Moreover define

$$M \supset A := K(G_1,1) \sqcup K(H,1) \times [0, 2/3[\Big/ \sim$$

$$M \supset B := K(H,1) \times]1/3, 1] \sqcup K(G_2,1) \Big/ \sim$$

   Note that M is obviously connected. We will separate the objective in two parts.

— $\underline{\pi_1(M) \cong G_1 *_H G_2}$: Note that A and B are open and their intersection $C := A \cap B = K(H,1) \times]1/3, 2/3[\simeq K(H,1)$ is connected. so we can apply Seifert Van Kampen and we get that

$$\pi_1(M) \cong \pi_1(K(G_1,1)) *_{\pi_1(K(H,1))} \pi_1(K(G_2,1)) \cong G_1 *_H G_2$$

— $\underline{\pi_n(M) \cong 0 \text{ for } n \geq 2}$: Let $\widetilde{M}$ be the universal cover of M and let $\widetilde{X_i}$ and $\widetilde{X_H}$ be the inverse images of $K(G_i,1)$ and $K(H,1)$ in $\widetilde{M}$. Since $K(G_i,1)$ and $K(H,1)$ have acyclic universal covers EG_i and EH, we can apply part 1) of this exercise to obtain that $\widetilde{X_i}, \widetilde{X_H}$ are also acyclic. Now, we apply the Mayer-Vietoris long exact sequence on the triad $(\widetilde{M} = \widetilde{X_1} \cup \widetilde{X_2}, \widetilde{X_1}, \widetilde{X_2})$ (observe that we have $\widetilde{X_1} \cap \widetilde{X_2} = \widetilde{X_H}$) :

$$... \to H_{n+1}(\widetilde{M}) \to H_n(\widetilde{X_H}) \to H_n(\widetilde{X_1}) \oplus H_n(\widetilde{X_2}) \to H_n(\widetilde{M}) \to ...$$

By the discussion above, $H_n(\widetilde{M})$ is then trivial for each $n \geq 2$. Now, it remains to show that it induces $\pi_n(M) \cong 0$ for $n > 2$. This actually follows from Hurewicz Theorem : since $\pi_1(\widetilde{M}) \cong 0$, we obtain $\pi_2(\widetilde{M}) \cong H_2(\widetilde{M}) \cong 0$. This allows to apply Hurewicz Theorem again and again to obtain

$$\pi_n(\widetilde{M}) \cong H_n(\widetilde{M}) \cong 0 \quad \forall n \geq 2.$$

We finally deduce

$$\pi_n(M) \cong \pi_n(\widetilde{M}) \cong 0 \quad \forall n \geq 2.$$

Hence, M is indeed a $K(G_1 *_H G_2, 1)$ space.

Exercice 5. Free action on even spheres.

Prove that the only non-trivial group acting freely on an even dimensional sphere is C_2.

Solution 5

By Lefschetz fixed point theorem, since we assume that G acts freely, one has for any $g \neq 1 \in G$:

$$0 = \sum_{i=0}^{n} (-1)^i \mathsf{Tr}(g_* \mid H_i(S^n, \mathbb{Z})).$$

Since S^n has non-trivial homology only in degree 0 and n, we get

$$\mathsf{Tr}(g_* \mid H_0(S^n, \mathbb{Z})) = -\mathsf{Tr}(g_* \mid H_n(S^n, \mathbb{Z}))$$

But g_* is the identity on H_0 and it is the multiplication by the degree on H_n. We get that any $g \neq 1 \in G$ acts with degree -1.

By compatibilities with composition, this forces G to have a single non trivial element, hence $G \simeq C_2$.